SELECTED FORMULAE

Coordinate transformations

Cylindrical $\mapsto$ rectangular	$(x,\, y,\, z) = (r\cos\phi,\, r\sin\phi,\, z)$
Spherical $\mapsto$ rectangular	$(x,\, y,\, z) = (r\sin\theta\cos\phi,\, r\sin\theta\sin\phi,\, r\cos\theta)$

Differential operators

Gradient
$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}$$

Divergence
$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \text{(Rectangular)}$$
$$\nabla \cdot \mathbf{F} = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,F_r) + \frac{1}{r}\,\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad \text{(Cylindrical)}$$

Curl
$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Line integral

Path ℓ
$$\int_\ell \mathbf{F}\cdot d\mathbf{l} = \int_0^1 \mathbf{F}(\mathbf{l}(s))\cdot\frac{d\mathbf{l}(s)}{ds}\,ds$$

Surface integrals

Rectangular surface ($z = 0$ plane)
$$\iint_A \mathbf{F}\cdot d\mathbf{A} = \int_a^b \int_c^d \mathbf{F}(x,y,0)\cdot(dx\,dy\,\hat{\mathbf{z}})$$

Circular surface ($z = 0$ plane)
$$\iint_A \mathbf{F}\cdot d\mathbf{A} = \int_0^R \int_0^{2\pi} \mathbf{F}(r,\phi,0)\cdot(r\,d\phi\,dr\,\hat{\mathbf{z}})$$

Cylindrical surface (no end faces)
$$\iint_A \mathbf{F}\cdot d\mathbf{A} = \int_0^L \int_0^{2\pi} \mathbf{F}(R,\phi,z)\cdot(R\,d\phi\,dz\,\hat{\mathbf{r}})$$

Spherical surface
$$\iint_A \mathbf{F}\cdot d\mathbf{A} = \int_0^{2\pi} \int_0^{\pi} \mathbf{F}(R,\phi,\theta)\cdot\left(R^2\,\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}\right)$$

Volume integrals

Cube
$$\iiint_v \rho\,dv = \int_0^a \int_0^a \int_0^a \rho(x,y,z)\,dx\,dy\,dz$$

Cylinder
$$\iiint_v \rho\,dv = \int_0^L \int_0^R \int_0^{2\pi} \rho(r,\phi,z)\,r\,d\phi\,dr\,dz$$

Sphere
$$\iiint_v \rho\,dv = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(r,\phi,\theta)\,r^2\,\sin\theta\,d\theta\,d\phi\,dr$$

Flux densities, fields and forces

Electric flux density and field	$\mathbf{D} = \epsilon\, \mathbf{E}$
Magnetic flux density and field	$\mathbf{B} = \mu\, \mathbf{H}$
Electric force and field	$\mathbf{F} = q\, \mathbf{E}$
Magnetic force and flux density	$\mathbf{F} = q\, \mathbf{v} \times \mathbf{B}$

Maxwell's Equations

Gauss (Electric)	$\nabla \cdot \mathbf{D} = \rho$	$\iint_A \mathbf{D} \cdot d\mathbf{A} = \iiint_{\mathsf{vol}(A)} \rho\, dv$
Gauss (Magnetic)	$\nabla \cdot \mathbf{B} = 0$	$\iint_A \mathbf{B} \cdot d\mathbf{A} = 0$
Faraday	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_{\ell(A)} \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t} \iint_A \mathbf{B} \cdot d\mathbf{A}$
Ampere	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_{\ell(A)} \mathbf{H} \cdot d\mathbf{l} = \iint_A \mathbf{J} \cdot d\mathbf{A} + \dfrac{\partial}{\partial t} \iint_A \mathbf{D} \cdot d\mathbf{A}$

Conservation Laws

Energy	$\mathbf{E} = -\nabla V$	$V = -\int_\ell \mathbf{E} \cdot d\mathbf{l}$
Charge	$\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$	$\iint_{\mathsf{area}(v)} \mathbf{J} \cdot d\mathbf{A} = -\dfrac{\partial}{\partial t} \iiint_v \rho\, dv$

Conductors

Current	$i = \iint_A \mathbf{J} \cdot d\mathbf{A}$	
Voltage	$v = -\int_\ell \mathbf{E} \cdot d\mathbf{l}$	
Ohm's Law	$\mathbf{J} = \sigma\, \mathbf{E}$	$v = R\, i$
Skin depth	$\delta = \sqrt{\dfrac{2}{\omega\, \mu\, \sigma}}$	
Permittivity (free space)	$\epsilon_\circ = 8.85 \times 10^{-12} \quad \mathsf{F\, m^{-1}}$	
Permeability (free-space)	$\mu_\circ = 4\,\pi \times 10^{-7} \quad \mathsf{H\, m^{-1}}$	
Conductivity (copper)	$\sigma = 5.8 \times 10^{7} \quad (\Omega\, \mathsf{m})^{-1}$	
Permittivity (copper)	$\epsilon = \epsilon_\circ = 8.85 \times 10^{-12} \quad \mathsf{F\, m^{-1}}$	
Permeability (copper)	$\mu = \mu_\circ = 4\,\pi \times 10^{-7} \quad \mathsf{H\, m^{-1}}$	
Charge redistribution time	$T_{\mathsf{CR}} = \dfrac{\epsilon}{\sigma}$	

Conductors (continued)

Resistance

$$R = -\frac{\int_\ell \mathbf{E} \cdot d\mathbf{l}}{\iint_A \sigma \, \mathbf{E} \cdot d\mathbf{A}}$$

$$R = \int_0^L \frac{dx}{\sigma(x)\, A(x)}$$

$$R = \frac{L}{\sigma\, A}$$

Capacitance (linear)

$$C = -\frac{\epsilon \iiint_v \nabla \cdot \mathbf{E} \, dv}{\int_\ell \mathbf{E} \cdot d\mathbf{l}}$$

$$C = \frac{\epsilon\, A}{d}$$

Divergence in cylindrical coordinates

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r\, F_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

A possibly useful formula

$$\frac{1}{b^2 - z^2} = \frac{1}{2b}\left[\frac{1}{b-z} + \frac{1}{b+z}\right]$$

Conversion between Cartesian, cylindrical, and spherical coordinates

		From		
		Cartesian	**Cylindrical**	**Spherical**
To	**Cartesian**	N/A	$x = \rho\cos\varphi$ $y = \rho\sin\varphi$ $z = z$	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$
	Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$	N/A	$\rho = r\sin\theta$ $\varphi = \varphi$ $z = r\cos\theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arccos\left(\frac{z}{r}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *destination* coordinates

	Cartesian	**Cylindrical**	**Spherical**
Cartesian	N/A	$\hat{\mathbf{x}} = \cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\varphi\hat{\boldsymbol{\rho}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Spherical	$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)z - \left(x^2 + y^2\right)\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$	$\hat{\mathbf{r}} = \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{z\hat{\boldsymbol{\rho}} - \rho\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *source* coordinates

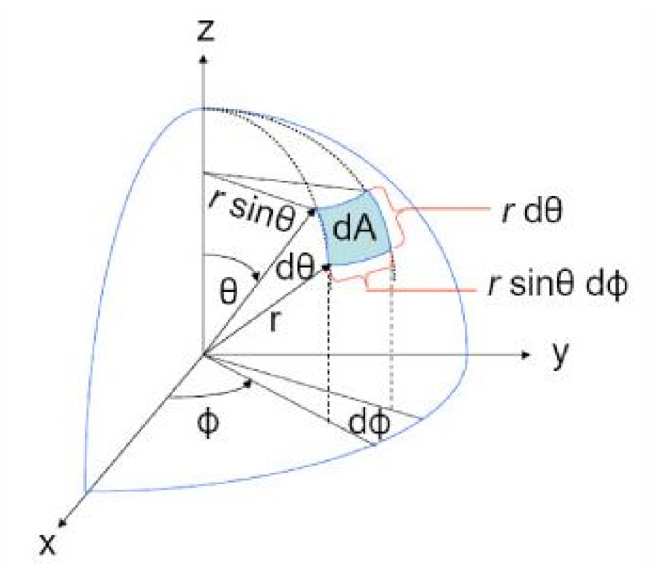
	Cartesian	**Cylindrical**	**Spherical**
Cartesian	N/A	$\hat{\mathbf{x}} = \frac{x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{y}} = \frac{y\hat{\boldsymbol{\rho}} + x\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \frac{x\left(\sqrt{x^2 + y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2 + y^2 + z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}$ $\hat{\mathbf{y}} = \frac{y\left(\sqrt{x^2 + y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2 + y^2 + z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \sqrt{x^2 + y^2}\hat{\boldsymbol{\theta}}}{\sqrt{x^2 + y^2 + z^2}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \frac{\rho\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \rho\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2 + z^2}}$
Spherical	$\hat{\mathbf{r}} = \sin\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol{\rho}} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is azimuthal
A vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\mathbf{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}}$ $+\frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}}$ $+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	— View by clicking [show] — [show]	— View by clicking [show] — [show]
Material derivative[α][1] $(\mathbf{A}\cdot\nabla)\mathbf{B}$	$\mathbf{A}\cdot\nabla B_x\hat{\mathbf{x}} + \mathbf{A}\cdot\nabla B_y\hat{\mathbf{y}} + \mathbf{A}\cdot\nabla B_z\hat{\mathbf{z}}$	— View by clicking [show] — [show]	— View by clicking [show] — [show]
Tensor divergence $\nabla\cdot\mathbf{T}$	— View by clicking [show] — [show]	— View by clicking [show] — [show]	— View by clicking [show] — [show]
Differential displacement $d\boldsymbol{\ell}$	$dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$	$d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}}$	$dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}}$
Differential normal area $d\mathbf{S}$	$dy\,dz\,\hat{\mathbf{x}}$ $+\,dx\,dz\,\hat{\mathbf{y}}$ $+\,dx\,dy\,\hat{\mathbf{z}}$	$\rho\,d\varphi\,dz\,\hat{\boldsymbol{\rho}}$ $+\,d\rho\,dz\,\hat{\boldsymbol{\varphi}}$ $+\,\rho\,d\rho\,d\varphi\,\hat{\mathbf{z}}$	$r^2\sin\theta\,d\theta\,d\varphi\,\hat{\mathbf{r}}$ $+\,r\sin\theta\,dr\,d\varphi\,\hat{\boldsymbol{\theta}}$ $+\,r\,dr\,d\theta\,\hat{\boldsymbol{\varphi}}$
Differential volume dV	$dx\,dy\,dz$	$\rho\,d\rho\,d\varphi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\varphi$

Non-trivial calculation rules [edit]

1. $\operatorname{div}\,\operatorname{grad} f \equiv \nabla\cdot\nabla f \equiv \nabla^2 f$
2. $\operatorname{curl}\,\operatorname{grad} f \equiv \nabla\times\nabla f = \mathbf{0}$
3. $\operatorname{div}\,\operatorname{curl}\mathbf{A} \equiv \nabla\cdot(\nabla\times\mathbf{A}) = 0$
4. $\operatorname{curl}\,\operatorname{curl}\mathbf{A} \equiv \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ (Lagrange's formula for del)
5. $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$



$$dA = \sqrt{r^4\sin^2(\theta)}d\theta d\phi = r^2\sin(\theta)d\theta d\phi$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Periodic Formulas

If n is an integer.

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas **(alternate form)**

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Cofunction Formulas

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

Math Formulas: Hyperbolic functions

Definitions of hyperbolic functions

1. $$\sinh x = \frac{e^x - e^{-x}}{2}$$

2. $$\cosh x = \frac{e^x + e^{-x}}{2}$$

3. $$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

4. $$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x}$$

5. $$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}$$

6. $$\coth\ x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x}$$

Derivatives

7. $$\frac{d}{dx}\ \sinh x = \cosh x$$

8. $$\frac{d}{dx}\ \cosh x = \sinh x$$

9. $$\frac{d}{dx}\ \tanh x = \operatorname{sech}^2 x$$

10. $$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \cdot \coth x$$

11. $$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$$

12. $$\frac{d}{dx}\ \coth x = -\operatorname{csch}^2 x$$

Hyperbolic identities

13. $$\cosh^2 x - \sinh^2 x = 1$$

14. $$\tanh^2 x + \operatorname{sech}^2 x = 1$$

15. $$\coth^2 x - \operatorname{csch}^2 x = 1$$

16. $$\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$$

17. $$\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$$

18. $$\sinh(2 \cdot x) = 2 \cdot \sinh x \cdot \cosh x$$

19. $$\cosh(2 \cdot x) = \cosh^2 x + \sinh^2 x$$

20. $$\sinh^2 x = \frac{-1 + \cosh 2x}{2}$$

21. $$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

Inverse Hyperbolic functions

22. $$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in (-\infty, \infty)$$

23. $$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1, \infty)$$

24. $$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad x \in (-1, 1)$$

25. $$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad x \in (-\infty, -1) \cup (1, \infty)$$

26. $$\text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad x \in (0, 1]$$

27. $$\text{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{|x|}\right), \quad x \in (-\infty, 0) \cup (0, \infty)$$

Derivatives of Inverse Hyperbolic functions

28. $$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$$

29. $$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

30. $$\frac{d}{dx}tanh^{-1} x = \frac{1}{1-x^2}$$

31. $$\frac{d}{dx}\text{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}$$

32. $$\frac{d}{dx}\text{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$$

33. $$\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}$$

Formulas from Calculus

Derivatives

$\frac{d}{dx}[x^n]$	$=$	nx^{n-1}	$\frac{d}{dx}[e^x]$	$=$	e^x	$\frac{d}{dx}[\sin x]$	$=$	$\cos x$
$\frac{d}{dx}[c]$	$=$	0	$\frac{d}{dx}[b^x]$	$=$	$b^x \ln b$	$\frac{d}{dx}[\cos x]$	$=$	$-\sin x$
$\frac{d}{dx}[x]$	$=$	1	$\frac{d}{dx}[\ln x]$	$=$	$\frac{1}{x}$	$\frac{d}{dx}[\tan x]$	$=$	$\sec^2 x$
$\frac{d}{dx}\left[\frac{1}{x}\right]$	$=$	$-\frac{1}{x^2}$	$\frac{d}{dx}[\log_b x]$	$=$	$\frac{1}{x \ln b}$	$\frac{d}{dx}[\sec x]$	$=$	$\tan x \sec x$
$\frac{d}{dx}\left[\frac{1}{x^2}\right]$	$=$	$-\frac{2}{x^3}$	$\frac{d}{dx}[\sinh x]$	$=$	$\cosh x$	$\frac{d}{dx}[\arcsin x]$	$=$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}[\sqrt{x}]$	$=$	$\frac{1}{2\sqrt{x}}$	$\frac{d}{dx}[\cosh x]$	$=$	$\sinh x$	$\frac{d}{dx}[\arctan x]$	$=$	$\frac{1}{1+x^2}$
$\frac{d}{dx}\left[\frac{1}{\sqrt{x}}\right]$	$=$	$-\frac{1}{2x\sqrt{x}}$	$\frac{d}{dx}[\tanh x]$	$=$	$\text{sech}\ ^2 x$			
			$\frac{d}{dx}[\text{arcsinh}\ x]$	$=$	$\frac{1}{\sqrt{1+x^2}}$			
			$\frac{d}{dx}[\text{arctanh}\ x]$	$=$	$\frac{1}{1-x^2}$			

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Chain Rule: $$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Special Cases

$$\frac{d}{dx}[f(x)^n] = nf(x)^{n-1}f'(x)$$

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \frac{-g'(x)}{g(x)^2}$$

$$\frac{d}{dx}[\ln|f(x)|] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$$

Integrals

$$\int x^n\ dx = \tfrac{1}{n+1}x^{n+1} + C \qquad \int e^x\ dx = e^x + C$$

$$\int \frac{1}{x}\ dx = \ln|x| + C \qquad \int b^x\ dx = \tfrac{1}{\ln b}b^x$$

$$\int c\ dx = cx + C \qquad \int \sinh x\ dx = \cosh x + C$$

$$\int x\ dx = \tfrac{1}{2}x^2 + C \qquad \int \cosh x\ dx = \sinh x + C$$

$$\int x^2\ dx = \tfrac{1}{3}x^3 + C \qquad \int \sin x\ dx = -\cos x + C$$

$$\int \frac{1}{x^2}\ dx = -\frac{1}{x} + C \qquad \int \cos x\ dx = \sin x + C$$

$$\int \sqrt{x}\ dx = \tfrac{2}{3}x\sqrt{x} + C \qquad \int \tan x\ dx = \ln|\sec x| + C$$

$$\int \frac{1}{\sqrt{x}}\ dx = 2\sqrt{x} + C \qquad \int \sec x\ dx = \ln|\tan x + \sec x| + C$$

$$\int \frac{1}{1+x^2}\ dx = \arctan x + C \qquad \int \sin^2 x\ dx = \tfrac{1}{2}(x - \sin x \cos x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + C \qquad \int \cos^2 x\ dx = \tfrac{1}{2}(x + \sin x \cos x) + C$$

$$\int \ln x\ dx = x\ln x\ - x + C \qquad \int \tan^2 x\ dx = \tan x\ - x + C$$

$$\int x^n \ln x\ dx = \tfrac{x^{n+1}}{n+1}\ln x\ - \tfrac{x^{n+1}}{(n+1)^2} + C \qquad \int \sec^2 x\ dx = \tan x + C$$

Substitution $\displaystyle \int f(g(x))g'(x)\ dx = \int f(u)\ du = F(u) + C = F(g(x)) + C$

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Special cases $\displaystyle \int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + C$

$$\int e^{f(x)}f'(x)\ dx = e^{f(x)} + C$$

By parts $\displaystyle \int u\ dv = uv - \int v\ du$

$$\int_a^b u\ dv = uv]_a^b - \int_a^b v\ du$$

or $\displaystyle \int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)\ dx$

$$\int_a^b f(x)g'(x)dx = f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\ dx$$

1 Resistors

$\boldsymbol{J}$ | Current Density
$\boldsymbol{v}$ | Velocity
$\boldsymbol{E}$ | Electric Field
σ | Conductivity
ρ | Resistivity
μ | Permeability
q | Charge per carrier
n | Carrier density
m | Carrier mass
τ | Time between collisions
μ_q | Mobility (ease with which a charge carrier can drift)

Equation for mobility

$$\mu_q = \frac{q\tau}{m}$$

Conductivity (dependent on charge, concentration and mobility)

$$\sigma = qn\mu_q$$

Essentially R=V/I

$$R = -\frac{\int_l \boldsymbol{E} \cdot d\boldsymbol{l}}{\iint_A \sigma \boldsymbol{E} \cdot d\boldsymbol{A}}$$

Volume integration over resistor

$$R = \int_0^L \frac{dx}{\sigma(x)A(x)}$$

Simplified version of volume integration for constant values

$$R = \frac{L}{\sigma A} = \frac{\rho L}{A}$$

Current density due to charge motion

$$\boldsymbol{J} = qn\boldsymbol{v}$$

Average charge velocity in resistor accounting for collisions

$$\boldsymbol{v} = (\frac{q\tau}{m})\boldsymbol{E}$$

Put above two together to get vector form of Ohm's Law

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\sigma = \frac{q^2\tau n}{m}$$

Skin Depth, the effective cross sectional area

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Resistance of conductor skin

$$R(\delta) = \frac{l}{\sigma A(\delta)}$$

2 Conductors and Insulators

ϵ | Permittivity
σ | Conductivity

Low frequency behaviour is dominated by σ, high frequency behaviour determined by how long charge takes to redistribute in the material, which is a material dependent property

$$T_{CR} = \frac{\epsilon}{\sigma}$$

Charge redistribution bandwidth is the corner frequency between being a conductor and insulator

$$B_{CR} = \frac{1}{2\pi T_{CR}} = \frac{\sigma}{2\pi\epsilon}$$

$$f << B_{CR} \text{ | Conductor}$$

$$f >> B_{CR} \text{ | Insulator}$$

Skin depth model used when it is significant, large conductivity means large current that generates charge separation which creates opposing electric field that attenuates the original field

$$\omega << \omega_{CR}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Charge redistribution model can be derived from the continuity equation

$$\frac{\delta\rho}{\delta t} + \frac{\sigma}{\epsilon}\rho = 0$$

3 Capacitors and Inductors

Φ | Magnetic Flux
n | Turns per unit length
ρ_l | Charge per unit length
$'h$ | Wire separation from central axis
a | Wire radius

Definition of Capacitance - charge separation per volt

$$C(v) = \frac{dQ(v)}{dv}$$

Assuming linearity from the above

$$C = \frac{\epsilon \iiint_v \nabla \cdot \boldsymbol{E} dv}{\int_l \boldsymbol{E} \cdot d\boldsymbol{l}}$$

Definition of Inductance

$$Li = \Phi$$

Self Inductance

$$L(i) = \frac{d\Phi}{di}$$

Coil Inductance

$$\hat{L} = \mu n^2 A$$

Infinite parallel wire capacitance

$$\hat{C} = \frac{\pi\epsilon}{\log(\frac{h}{a} + \sqrt{(\frac{h}{a})^2 - 1})}$$

Two parallel wires

$$V = \frac{\rho_l}{\pi\epsilon}\log(\frac{h}{a} + \sqrt{(\frac{h}{a})^2 - 1})$$

$$\hat{L} = \frac{\mu}{\pi}\log(\frac{h-a}{a})$$

$$\hat{C} = \frac{\pi\epsilon}{\log(\frac{h}{a})}$$

Wire above plane

$$\hat{L} = \frac{\mu}{2\pi}\log(\frac{2h}{a})$$

$$\hat{C} = \frac{2\pi\epsilon}{\log(\frac{2h}{a})}$$

Coaxial cable

$$\hat{L} = \frac{\mu}{2\pi}\log(\frac{b}{a})$$

$$\hat{C} = \frac{2\pi\epsilon}{\log(\frac{b}{a})}$$

4 Wires

Non-ideal properties

- Signal Distortion
  - Ringing response of conductors at high frequencies is due to inherent inductance and capacitance of wires (skin depth)
  - Wires act as a low pass filter
  - Affected by resistance, self-inductance and capacitance
  - Mitigated by
    * Slower rise times, meaning a lower knee frequency so fknee ¡¡ fring
    * Shorter wire lengths, shifts the resonant behaviour to higher frequencies
    * Flattening high frequency impedance (achieved through zero reflection coefficient)
- EMI
  - Electromagnetic radiation caused by quickly changing currents and voltages through conductors in high-speed digital systems
  - Amperes Law broadcast your digital signal wirelessly by accident
  - Caused by mutual inductance and capacitance
  - Big problem when there are current loops
    * Current flowing out a pin, along a wire, into another device, and back via a ground plane or wire
    * The area enclosed by the loop is the problem
    * Behave like antennas
  - Mitigated by
    * Increasing rise time
    * Keep current loops in small area e.g coax cable
    * Shielding
- Cross Talk
  - Induced voltages and currents due to EMI, causes noise
  - Faradays Law receive digital signals wirelessly by accident
  - Caused by mutual inductance and capacitance
  - Increased by a faster rise time, both in inductive and capacitive cross-talk
  - Mitigated in same way as EMI

Consequences

- Ground bounce
  - Connection between internal and system ground is inductive, so due to $L = \frac{di}{dt}$ a large change in current will change ground voltage
  - Mitigated by
    * Lower inductance packaging
    * Larger diameter ground wires
    * Ground wires closer to ground plane
    * Edge slowing
    * Lower voltage family
    * Separate input group reference
    * More ground wires

h	Separation from center
a	Smaller radius
b	Larger radius
ν	Signal propagation velocity
D	Propagation delay per unit length
t_{SW}	Switching time

Two parallel wires

$$\hat{L} = \frac{\mu}{\pi} \log(\frac{h-a}{a})$$

$$\hat{C} = \frac{\pi\epsilon}{\log(\frac{h}{a})}$$

Wire above plane

$$\hat{L} = \frac{\mu}{2\pi} \log(\frac{2h}{a})$$

$$\hat{C} = \frac{2\pi\epsilon}{\log(\frac{2h}{a})}$$

Coaxial cable

$$\hat{L} = \frac{\mu}{2\pi} \log(\frac{b}{a})$$

$$\hat{C} = \frac{2\pi\epsilon}{\log(\frac{b}{a})}$$

Signal propagation velocity

$$\nu = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$$

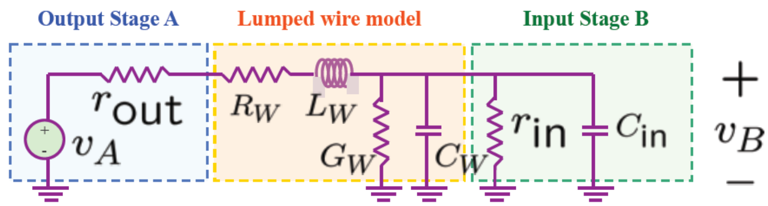
Propagation delay per unit length

$$D = \frac{1}{\nu} = \sqrt{\hat{L}\hat{C}} = \sqrt{\mu\epsilon}$$

Rising edge length

$$l_{sw} = \frac{t_{sw}}{D}$$

4.a Lumped Model



Y	Voltage over inductor
v_B	Voltage on receiving system side of capacitance

Lumped model occurs when $l_{SW} >> l$
Knee frequency characterizes the approximate bandwidth of digital signals

$$f_{KNEE} = \frac{1}{\pi t_{SW}} \approx \frac{0.5}{t_{SW}}$$

Interconnect resonant frequency, signal will be distorted as the knee frequency exceeds the ringing frequency

$$f_{RING} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}} \approx \frac{1}{2\pi\sqrt{LC}}$$

Inductive Cross Talk

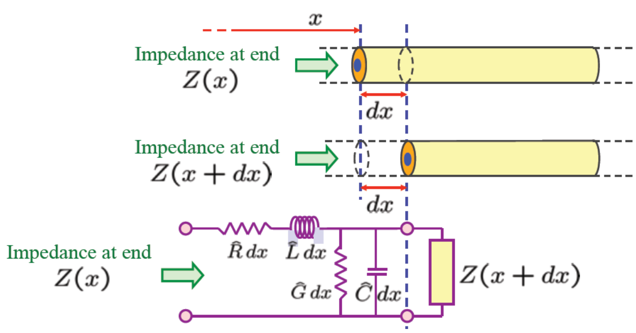
$$\frac{Y}{\Delta V} \approx \frac{L_M}{R_A t_{SW}} \approx \frac{L_M C_A}{t_{SW}^2}$$

Capacitive Cross Talk

$$\frac{v_B}{\Delta V} \approx \frac{R_B C_M}{t_{SW}}$$

4.b Distributed Model

G | Conductivity
ρ | Reflection Coefficient
Z_O | Characteristic Impedance
Γ| Attenuation Coefficient



Distributed model occurs when $l_{SW} << l$, these types of systems often suffer from signal distortion

$$\hat{Z}_{series} = \hat{R} + s\hat{L}$$

$$\hat{Y}_{shunt} = \hat{G} + s\hat{C}$$

Characteristic impedance is of an infinite length of wire

$$Z_O = \sqrt{\frac{\hat{Z}_{series}}{\hat{Y}_{shunt}}}$$

At low frequencies

$$Z_O(j\omega) = \sqrt{\frac{\hat{R}}{\hat{G}}}$$

At high frequencies (also the characteristic impedance of a lossless line)

$$Z_O(j\omega) = \sqrt{\frac{\hat{L}}{\hat{C}}}$$

Distributed impedance model for a finite length load terminated wire

$$\frac{dZ(x)}{dx} = \hat{Y}_{shunt}[Z(x)]^2 - \hat{Z}_{series}$$

$$Z(l) = Z_{load}$$

Resonant frequency

$$f_{ring} = \frac{1}{(4l)D}$$

Reflection Coefficient

$$\rho(s) = \frac{Z_{load}(s) - Z_O(s)}{Z_{load}(s) + Z_O(s)}$$

Attenuation Coefficient

$$\Gamma(s) = \sqrt{Z_{series}(s)Y_{shunt}(s)} = D_{lossless}s = s/v$$

Impedance for a finite length lossless wire

$$Z_{wire}(s) = Z_O(s)(\frac{1 + \rho e^{-2sl/v}}{1 - \rho e^{-2sl/v}})$$

Transfer function for output voltage/input voltage

$$H_{AB}(s) = \frac{(1 + \rho_B \exp(-sl/\nu)}{1 + \rho_B \exp(-2sl/\nu)}$$

5 Semiconductors

Band-gap model

- Pauli's exclusion principle means no two identical particles can occupy the same quantum state
- Conduction band contains free to move electron
- Valence band contains electrons bound to specific atoms (immobile)
- Band gap is higher for insulators, lower for conductors
- Extrinsic semiconductors are intrinsic semiconductors with a dopant added
  - n-type is with a donor level added below conduction band that makes it easier for electrons to jump up (Group V dopant)
  - p-type is with an acceptor level added above the valence band that makes it easier for holes to move in the valence band (mediated by electrons)

n	Mobile conduction band electrons
N_A	Immobile acceptor ions
p	Mobile valence band holes
N_D	Immobile donor ions
T	Temperature
α	Recombination proportionality constant

Charge density is given by

$$\rho = q(p + N_D - n - N_A)$$

Electrons and holes are in pairs (dependent on temperature), so the EHP generation rate is

$$r_{gen} = n_i(T)^2$$

Electrons and holes also recombine in pairs, so the EHP recombination rate is dependent on concentration of both particles

$$r_{rec} = np$$

Electrons and holes can also be injected in via diffusion or contact with another conductor, therefore the mobile electron concentration model is

$$\frac{dn(t)}{dt} = \alpha[n_i^2 - n(t)p(t)] + r_{in}(t)$$

Under equilibrium conditions with no injection

$$\frac{dn(t)}{dt} = 0 = \frac{dp(t)}{dt}$$

$$\bar{n}\bar{p} = n_i^2$$

$$\bar{n} - \bar{p} = N_D - N_A$$

Therefore the concentrations are

$$\bar{n} = \frac{1}{2}(N_D - N_A) + \sqrt{\frac{1}{4}(N_A - N_D)^2 + n_i(T)^2}$$

For an n-type semiconductor

$$N_D >> n_i^2,\ N_A = 0$$

$$\bar{n} \approx N_D$$

$$\bar{p} \approx \frac{n_i^2}{N_D}$$

For a p-type semiconductor

$$N_A >> n_i^2,\ N_D = 0$$

$$\bar{p} \approx N_A$$

$$\bar{n} \approx \frac{n_i^2}{N_A}$$

5.a Currents in semiconductors

L_p | Diffusion length for holes
L_n | Diffusion length for electrons
D_p | Diffusion coeffient for holes
D_n | Diffusion coefficient for electrons
Δp | Hole perturbation at injection
Δn | Electron perturbation at injection

Current density in the semiconductor is due to drift and diffusion

$$\boldsymbol{J}_n = qn\mu_n\boldsymbol{E} + qD_n\nabla n$$

$$\boldsymbol{J}_p = qn\mu_p\boldsymbol{E} + qD_p\nabla p$$

$$\boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_p$$

Where diffusion coefficients are

$$D_p = \frac{kT}{q}\mu_p$$

$$D_n = \frac{kT}{q}\mu_n$$

Recombination lifetime

$$\tau_r = \frac{1}{\alpha(\bar{n}+\bar{p})}$$

For p-type and n-type respectively, due to dominant carriers, the recombination rate is

$$\tau_r \approx \tau_p = \frac{1}{\alpha\bar{p}}$$

$$\tau_r \approx \tau_n = \frac{1}{\alpha\bar{n}}$$

If space charge neutrality is assumed, a perturbation in one carrier will instantly result in a perturbation in the other carrier, this is due to the 'instant' charge redistribution time compared to recombination, therefore

$$\rho = q(p + N_D - n - N_A) \text{ (from before)}$$

$$p(t) = n(t) + N_A - N_D$$

Define perturbations in the carriers as

$$\delta n(t,x) = n(t,x) - \bar{n}$$

$$\delta p(t,x) = p(t,x) - \bar{p}$$

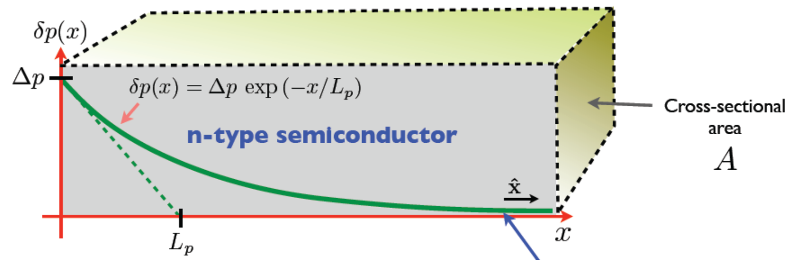
Yields a pair of continuity equations

$$\frac{1}{q}\nabla\cdot\boldsymbol{J}_p = -\frac{d\delta p}{dt} - \frac{\delta p}{\tau_p}$$

$$-\frac{1}{q}\nabla\cdot\boldsymbol{J}_n = -\frac{d\delta n}{dt} - \frac{\delta n}{\tau_n}$$

For the special case when there is no drift current, and all current is due to carrier diffusion, the diffusion current density due to electrons is

$$\boldsymbol{J}_n = qD_n\nabla n$$



The diffusion equations for electrons and holes are

$$\frac{d\delta n}{dt} = D_n\nabla^2(\delta n) - \frac{\delta n}{\tau_n}$$

$$\frac{d\delta p}{dt} = D_p\nabla^2(\delta p) - \frac{\delta p}{\tau_p}$$

Define the diffusion length of carriers as

$$L_p = \sqrt{D_p\tau_p}$$

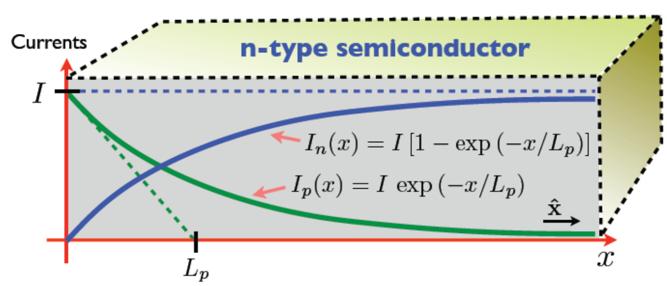
$$L_n = \sqrt{D_n\tau_n}$$

By solving with boundary conditions the excess carrier concentrations $\delta p,\ \delta n$ can be found

$$\delta p(0) = \Delta p$$

$$\delta p(\infty) = 0$$

$$\delta p(x) = \Delta p exp(-x/L_p)$$



The total current at any point flowing through the device is due to hole injection at $x = 0$, where electron current is zero

$$I = I_p(0) = (\frac{qAD_p}{L_p}\Delta p)$$

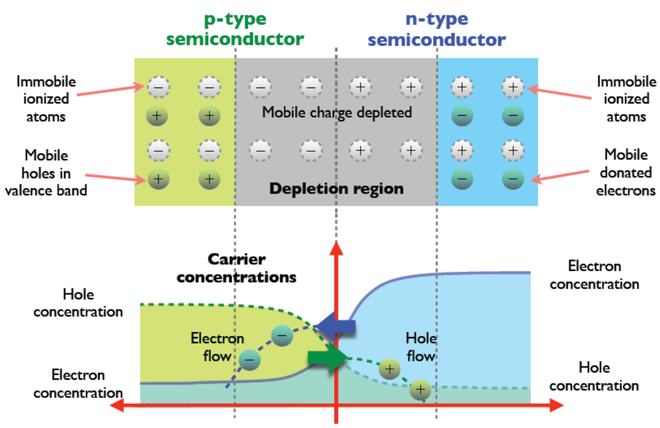
The electron current must pick up as hole current decays, the individual currents are

$$I_n(x) = I[1 - \exp(-x/L_p)]$$

$$I_p(x) = I\exp(-x/L_p)$$

$$I = I_n + I_p$$

5.b Junctions



- Carrier gradients between the two semiconductors leads to diffusion between the two
- Electrons diffuse from n to p
- Holes diffuse from p to n
- Mobile charge is depleted in central 'depletion region'
- Depletion approximation assumes that the entire voltage drop is over the depletion region
- Charge separation causes an electric field that opposes diffusion and creates equilibrium

D_p	Diffusion coeffient for holes
D_n	Diffusion coefficient for electrons
∇p	Gradient of p
∇n	Gradient of n
μ_p	Hole mobility
μ_n	Electron mobility
p	Mobile hole concentration
n	Mobile electron concentration
k	Boltzmann's constant
L_p	Diffusion length for holes
L_n	Diffusion length for electrons
Δp	Hole perturbation at injection
Δn	Electron perturbation at injection

At equilibrium, $\boldsymbol{J}_n = \boldsymbol{J}_p = 0$

$$0 = qn\mu_n\boldsymbol{E} + qD_n\nabla n$$

$$0 = qn\mu_p\boldsymbol{E} + qD_p\nabla p$$

Therefore the electric field at equilibrium is

$$\boldsymbol{E} = \frac{D_p}{\mu_p}(\frac{1}{p}\nabla p)$$

The "Einstein relation"

$$\frac{D_p}{\mu_p} = \frac{kT}{q}$$

Integrate over a path through this electric field with initial conditions of the injected carrier concentrations to find contact potential

$$V_O = \frac{D_p}{\mu_p}\log(\frac{p_p}{p_n}) = \frac{kT}{q}\log(\frac{p_p}{p_n}) \approx \frac{kT}{q}\log(\frac{N_A N_D}{n_i^2})$$

With an applied voltage V

$$V_O - V = \frac{D_p}{\mu_p}\log(\frac{p_p^V}{p_n^V})$$

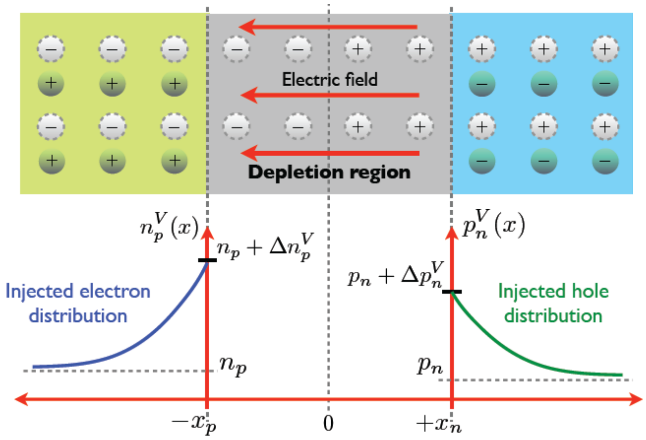
Solving for the new carrier concentrations

$$\frac{p_n^V}{p_p^V}\frac{p_p}{p_n} = \exp(\frac{qV}{kT})$$

$$p_p^V \approx p_p$$

$$p_n^V \approx p_n \exp(\frac{qV}{kT})$$

$$n_p^V \approx n_p \exp(\frac{qV}{kT})$$



Applying this to solve for the injected carrier distribution

$$\Delta p_n^V = p_n(\exp(\frac{qV}{kT}) - 1)$$

$$\Delta n_p^V = n_p(\exp(\frac{qV}{kT}) - 1)$$

Integrate to find the diffusion currents due to injected hole/electron distributions in n/p-type materials

$$I_p(x) = (\frac{qAD_p}{L_p})\Delta p_n^V \exp(-\frac{x - x_n}{L_p}), \ x \geq x_n$$

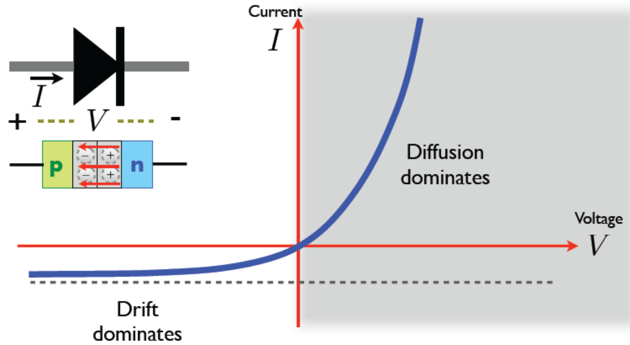
$$I_n(x) = (\frac{qAD_n}{L_n})\Delta n_p^V \exp(-\frac{x + x_p}{L_n}), \ x \geq x_n$$

Shockley Diode Equation can be obtained by summing the two currents

$$I = I_O[\exp(\frac{qV}{kT}) - 1]$$

The saturation current is

$$I_O = qA(\frac{D_p p_n}{L_p} + \frac{D_n n_p}{L_n})$$



Forward bias (diffusion dominates)

$$V >> 0$$

$$I \approx I_O \exp(\frac{qV}{kT})$$

Reverse bias (drift dominates)

$$V << 0$$

$$I \approx -I_O$$

Conductivity of neutral regions can be used to determine if negligible voltage drop assumption is valid

$$\sigma_p = q(\mu_p \bar{p}_p + \mu_n \bar{n}_p) \approx q\mu_p \bar{p}_p$$

$$\sigma_n = q(\mu_p \bar{p}_n + \mu_n \bar{n}_n) \approx q\mu_n \bar{n}_n$$

General Assumptions

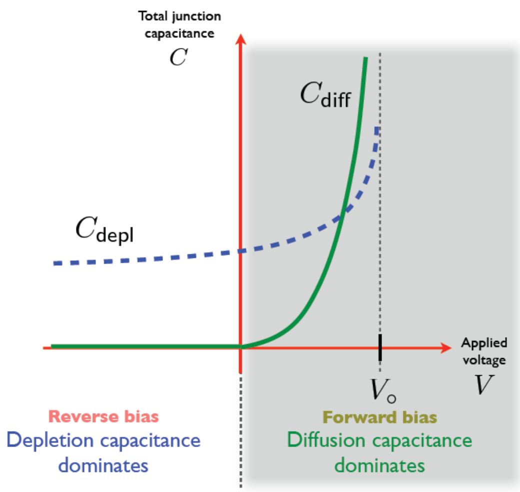
- Electric field is confined to the junction and there is no electric field in neutral regions
- One dimensional device

5.c Junction Dynamics

There are two charge storage mechanism in junctions, and therefore two capacitances

- Depletion region capacitance
- Diffusion capacitance

D_p	Diffusion coeffient for holes
p_n	Equilibrium minority carrier concentration (holes)
k	Boltzmann's constant
L_p	Diffusion length for holes
τ_p	Recombination lifetime (holes)



Depletion capacitance

$$q(t) = Q_{depl}(v(t))$$

$$C_{depl}(v) = \frac{dQ_{depl}}{dv}$$

$$C_{depl}(V) = \epsilon A \sqrt{\frac{q}{2\epsilon(V_O - V)\frac{N_D N_A}{N_D + N_A}}}$$

Diffusion capacitance (two expressions for mobile holes and mobile electrons)

$$q(t) = Q_{diff}(v(t))$$

$$C_{diff}(v) = \frac{dQ_{diff}}{dv}$$

$$C_{depl}(V) = \frac{q^2}{kT} A L_p p_n \exp(\frac{qV}{kT})$$

For a long p+n junction diode, we can assume junction operation is dominated by hole injection, and that all recombination happens before the end of the n-type material.

The total charge due to injected holes on the n-type region is:

$$Q_p(t) = \int_{x_n}^{X} qA\delta p_n(t, x)dx$$

Using this, the charge control model for both forward and reverse biased junctions can be derived

$$i(t) = \frac{dQ(t)}{dt} + \frac{Q(t)}{\tau}$$

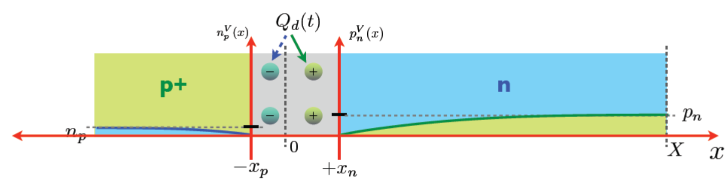
The charge storage delay for $p^+ - n$ and $p - n^+$ respectively

$$t_{CSD} = \tau_{p_n} \log(1 + \frac{I_f}{I_r})$$

$$t_{CSD} = \tau_{n_p} \log(1 + \frac{I_f}{I_r})$$

The charge control model can be applied for forward or reverse biased junctions.

- In forward bias, the charge is due to carrier injection in neutral regions either side of the depletion range. Separate models hold for both electrons and holes.
- In reverse bias, the charge is due to the "uncovered" ions in the depletion region



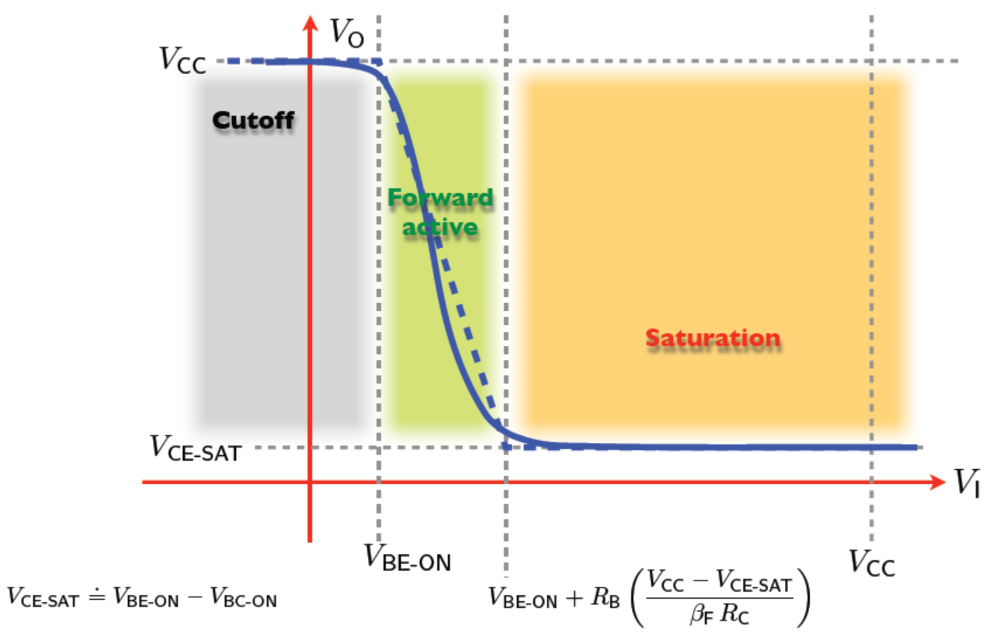
- Derivative term implies that changes in stored charge lag behind changes in current
- When the diode is "turned off"
  - Charge storage delay
    * The applied voltage switches sign
    * The excess of injected holes decays
    * Diode voltage drops from contact potential down to zero
  - Carrier depletion
    * Diode enters reverse bias
    * Depletion region expands
    * Diode voltage settles at -E
- For "turn on" the reverse
- Assumptions
  - n-type region is long $X >> L_p$
  - p-type region is heavily doped compared with n-type region $p_n >> n_p$
  - Half-period is much longer than recombination lifetime $T >> \tau_p$
  - Forward bias junction voltage is limited by contact potential $v(t) \leq V_O$
  - Contact potential is much smaller than magnitude of applied voltage $V_O << E$
  - Reverse bias saturation is very small $I_O \approx 0$

6 Bipolar Junction Transistors

Applied Voltages	B-E junction bias	B-C junction bias	Mode
$E < B < C$	Forward	Reverse	Forward-active
$E < B > C$	Forward	Forward	Saturation
$E > B < C$	Reverse	Reverse	Cut-off
$E > B > C$	Reverse	Forward	Reverse-active

- Forward Active
  1. Carrier injection - forward biased BE junction
     - Injection of minority electrons into the p-type base region
     - Injection of minority holes into the n-type emitter region
  2. Carrier diffusion - base transport
     - Diffusion of injected base electrons towards the collector
     - Recombination of some injected electrons with majority holes
     - Diffusion of remainder into the BC junction depletion region
  3. Carrier Drift - BC depletion region
     - Drift of minority electrons from base to collector
     - Conventional current flows into the collector

  - Large current gain due to minority electron current dominating BE current (BE junction is pn+)
  - Collector current smaller than emitter current due to recombination across base and emitter hole current
- Reverse Active
- Saturation
- Cut-off



β | Current Gain

Ebers-Moll model
Current transfer ratios

$$\alpha_R = \frac{B_R}{1+\beta_R}$$

$$\alpha_F = \frac{B_F}{1+\beta_F}$$

BE junction current

$$I_F = I_{ES}[\exp(\frac{qV_{BE}}{kT}) - 1]$$

BC junction current

$$I_R = I_{CS}[\exp(\frac{qV_{BC}}{kT}) - 1]$$

Total emitter current

$$I_E = I_F - \alpha_R I_R$$

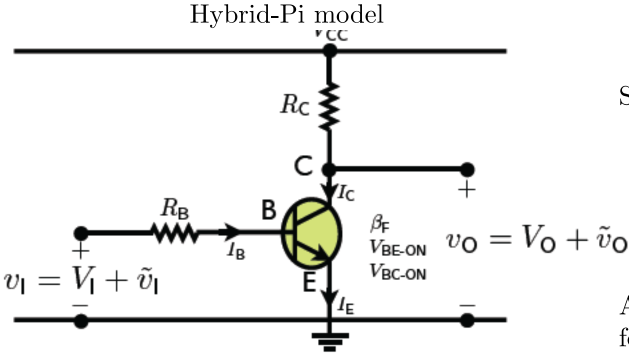
Total collector current

$$I_C = \alpha_F I_F - I_R$$

Total base current

$$I_B = I_E - I_C$$

Hybrid-Pi model



Local Gain

$$\frac{\tilde{v}_O}{\tilde{v}_I} \approx \frac{R_C\beta_F}{R_B + r_\pi}$$

6.a BJT Dynamics

Ebers-Moll and Hybrid-Pi are static models that ignore internal BJT dynamics, a dynamic model accounts for applied terminal voltages and currents that are time varying.
Steady state diffusion equation for electrons:

- Forward active
  - Approximately linear for narrow base widths
  - Charge injection is across BE junction only
  - Injected charge distribution dominates charge storage
- Cutoff
  - No injected charge
  - Charge stored in depletion region dipoles only
- Reverse active mode
  - Carrier injection across BC junction only
  - Injected charge dominates charge storage
- Saturation
  - Carrier injection across both junctions
  - Superposition of forward and reverse active moves
  - Injected charge dominates charge storage

$$\delta n(t,x) = n_p(t,x) - n_p$$

$$0 = D_N\frac{d^2\delta n_p(x)}{dx^2} - \frac{\delta n_p(x)}{\tau_n}$$

Solve using boundary conditions

$$\delta n_p(0) = \Delta n_E = n_p[\exp(\frac{qV_{BE}}{kT}) - 1]$$

$$\delta n_p(W_b) = \Delta n_C = -n_p$$

Assuming that the p-type base material has relatively few electrons compared to that injected by the emitter, the excess electron concentration in the base is

$$\delta n_p(x) = \Delta n_p(\frac{\exp(\frac{W_b-x}{L_n}) - \exp(-\frac{W_b-x}{L_n})}{\exp(\frac{W_b}{L_n}) - \exp(-\frac{W_b}{L_n})})$$

The terminal currents are therefore

$$I_E = qAD_n\frac{d\delta n_p(x)}{dx}|_{x=0}$$

$$I_C = qAD_n \frac{d\delta n_p(x)}{dx}|_{x=0}$$
$$I_B = I_E - I_C$$

q_F	Charge stored in forward active mode excess carrier distribution
q_R	Charge stored in reverse active mode excess carrier distribution
q_{BE}	Charge stored in BE junction depletion region
q_{BC}	Charge stored in BC junction depletion region
τ_F	Mean minority carrier transit time across base in forward active mode
τ_R	Mean minority carrier transit time across base in reverse active mode
τ_{BF}	Minority carrier lifetime (in base) in forward active mode
τ_{BR}	Minority carrier lifetime (in base) in reverse active mode

$i_C =$	$-\frac{q_F}{\tau_F}$	$+\frac{dq_{BC}}{dt}$	$+q_R\left(\frac{1}{\tau_R}+\frac{1}{\tau_{BR}}\right)+\frac{dq_R}{dt}$
$i_B =$	$-\frac{q_F}{\tau_{BF}}-\frac{dq_F}{dt}$	$-\frac{dq_{BE}}{dt}-\frac{dq_{BC}}{dt}$	$-\frac{q_R}{\tau_{BR}}-\frac{dq_R}{dt}$
$i_E =$	$-q_F\left(\frac{1}{\tau_F}+\frac{1}{\tau_{BF}}\right)-\frac{dq_F}{dt}$	$-\frac{dq_{BE}}{dt}$	$+\frac{q_R}{\tau_R}$
	Electron injection across BE junction	Depletion region capacitances	Electron injection across BC junction